SOLUTIONS OF CERTAIN DUAL INTEGRAL EQUATIONS

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In the process of solving certain problems of the theory of elasticity (hydromechanics, electrostatics, etc.), when the boundary conditions of the problem on a part of a boundary surface are given in one form while on the remaining part of this surface they are given in some other form, it is frequently advantageous to reduce the solution of the problem to the determination of an unknown function by means of dual integral equations. Such integral equations have been considered only for particular cases. For example, dual integral equations containing Bessel functions or trigonometeric functions were considered in the works of King [1], Busbridge [2], Noble [3 and 4] and others. Solutions of some such equations are also given in the works of Titchmarsh [5], Sneddon [6] and Tranter [7].

In the present paper there are considered certain integral equations containing Legendre functions with a complex index (conical functions) and also equations which contain trigonometric functions.. It seems to the author that such dual equations are being considered for the first time here.

The solutions of these equations are obtained formally by a single method presented in the work [8]. It is proved that the obtained solution is valid for one type of equations; but a similar proof can be given for the remaining equations. A problem in elasticity theory is treated as a practical example.

The following formulas will be used in the solution of the considered dual integral equations.

The integral representation of Legendre's function with a complex index [9] is

$$P_{-1/2+i\tau}(\cosh\alpha) = \frac{\sqrt{2}}{\pi}\int_0^\alpha \frac{\cos\tau s\,ds}{\sqrt{\cosh\alpha-\cosh s}}$$

$$P_{-1/2+i\tau}(\cosh\alpha) = \frac{\sqrt{2}}{\pi}\coth\pi\tau\int_\alpha^\infty \frac{\sin\tau s\,ds}{\sqrt{\cosh s-\cosh\alpha}} \qquad (0.1)$$

The Mehler-Fock [10] transform is given by

$$f(\alpha) = \int_0^\infty g(\tau) P_{-1/2+i\tau}(\cosh\alpha)\, d\tau \qquad (0 < \alpha < \infty) \tag{0.2}$$

$$g(\tau) = \tau \tanh \pi\tau \int_0^\infty f(\alpha) P_{-1/2+i\tau}(\cosh\alpha) \sinh\alpha \, d\alpha \qquad (\tau \geqslant 0)$$

The solution of Abel's integral equations are

$$\begin{aligned} f(x) &= \int_a^x \frac{u(\xi)\, d\xi}{(x-\xi)^\mu}, \qquad u(z) = \frac{\sin\mu\pi}{\pi} \frac{d}{dz} \int_a^z \frac{f(x)\, dx}{(z-x)^{1-\mu}} \\ f(x) &= \int_x^b \frac{u(\xi)\, d\xi}{(\xi - x)^\mu}, \qquad u(z) = -\frac{\sin\mu\pi}{\pi} \frac{d}{dz} \int_z^b \frac{f(x)\, dx}{(x-z)^{1-\mu}} \end{aligned} \tag{0.3}$$

We shall also use the values of the integrals [11]

$$\begin{aligned} \int_0^\infty P_{-1/2+i\tau}(\cosh\alpha) \cos(\tau s)\, d\tau &= \begin{cases} [2(\cosh\alpha - \cosh s)]^{-1/2} & (0 < s < \alpha) \\ 0 & (0 < \alpha < s) \end{cases} \\ \int_0^\infty \tanh \pi\tau P_{-1/2+i\tau}(\cosh\alpha) \sin(\tau s)\, d\tau &= \begin{cases} [2\cosh s - \cosh\alpha)] & (0 < \alpha < s) \\ 0 & (0 < s < \alpha) \end{cases} \end{aligned} \tag{0.4}$$

and equations

$$\begin{aligned} \sqrt{2} \cos\tau s &= \frac{d}{ds} \int_0^s \frac{P_{-1/2+i\tau}(\cosh\alpha) \sinh\alpha \, d\alpha}{\sqrt{\cosh s - \cosh\alpha}} \\ \sqrt{2} \sin\tau s &= -\tanh \pi\tau \frac{d}{ds} \int_s^\infty \frac{P_{-1/2+i\tau}(\cosh\alpha) \sinh\alpha \, d\alpha}{\sqrt{\cosh\alpha - \cosh s}} \\ \sqrt{2} \frac{\cos\tau s}{\tau} &= \tanh \pi\tau \int_s^\infty \frac{P_{-1/2+i\tau}(\cosh\alpha) \sinh\alpha \, d\alpha}{\sqrt{\cosh\alpha - \cosh s}} \\ \sqrt{2} \frac{\sin\tau s}{\tau} &= \int_0^s \frac{P_{-1/2+i\tau}(\cosh\alpha) \sinh\alpha \, d\alpha}{\sqrt{\cosh s - \cosh\alpha}} \end{aligned} \tag{0.5}$$

These last results can be obtained formally from (0.1) if one considers them as integral equations of the Abel type and makes use of the solutions (0.3); or one may derive them from (0.4) and (0.2).

Some of the integrals of (0.5) may possibly diverge, but formally one may use them for a quick derivation of general solutions of the dual integral equations considered here.

1. We consider the dual integral equations

$$\begin{aligned} \int_0^\infty f(\tau) P_{-1/2+i\tau}(\cosh\alpha)\, d\tau &= g(\alpha) \qquad (0 < \alpha < a) \\ \int_0^\infty \tau \tanh \pi\tau\, f(\tau) P_{-1/2+i\tau}(\cosh\alpha)\, d\tau &= h(\alpha) \qquad (a < \alpha < \infty) \end{aligned} \tag{1.1}$$

Let us multiply the first one of these equations by $\sinh\alpha\,(\cosh s - \cosh\alpha)^{-\frac{1}{2}}$, integrate the result with respect to α from 0 to s, and differentiate the obtained equation with respect to s. The second equation of (1.1) we multiply by $\sinh\alpha\,(\cosh\alpha - \cosh s)^{-\frac{1}{2}}$ and integrate with respect to α from s $(s > a)$ to ∞. Then, in view of (0.5), the system (1.1) can be reduced to the form

$$\sqrt{2}\int_0^\infty f(\tau)\cos\tau s\,d\tau = G'(0, s) \qquad (0 \leqslant s < a)$$

$$\sqrt{2}\int_0^\infty f(\tau)\cos\tau s\,d\tau = H(s, \infty) \qquad (a < s < \infty) \tag{1.2}$$

where

$$G(0, c) = \int_0^c \frac{g(\alpha)\sinh\alpha\,d\alpha}{\sqrt{\cosh s - \cosh\alpha}}, \qquad H(c, \infty) = \int_c^\infty \frac{h(\alpha)\sinh\alpha\,d\alpha}{\sqrt{\cosh\alpha - \cosh s}} \tag{1.3}$$

Making use of the inversion formula for the cosine Fourier transform, we obtain from (1.2)

$$\frac{\pi}{\sqrt{2}}\,f(\tau) = \int_0^a G'(0, s)\cos\tau s\,ds + \int_a^\infty H(s, \infty)\cos\tau s\,ds \tag{1.4}$$

One can prove that if the integrals in (1.4) exist, then the solution of the dual integral equations (1.1) can be expressed by means of Formula (1.4).

Let us evaluate the first one of the integrals (1.1) in the region $(a < \alpha < \infty)$. Substituting the value of $f(\tau)$ from (1.4) into this integral, and taking into account (0.4), we obtain

$$\pi\int_0^\infty f(\tau)\,P_{-1/2+i\tau}(\cosh\alpha)\,d\tau = \int_0^a \frac{G'(0, s)\,ds}{\sqrt{\cosh\alpha - \cosh s}} + \int_a^\alpha \frac{H(s, \infty)\,ds}{\sqrt{\cosh\alpha - \cosh s}} \tag{1.5}$$

Let us denote the value of the second integral of the system (1.1) in the region $(0 \leqslant \alpha < a)$ by $V(\alpha)$. Then

$$V(\alpha) = \int_0^\infty \tau\tanh\pi\tau f(\tau)\,P_{-1/2+i\tau}(\cosh\alpha)\,d\tau \qquad (0 \leqslant \alpha < a) \tag{1.6}$$

From the second equation of (1.1) and from (1.6) it follows, in view of (0.2), that

$$f(\tau) = \int_0^a V(\alpha)\,P_{-1/2+i\tau}(\cosh\alpha)\sinh\alpha\,d\alpha + \int_a^\infty h(\alpha)\,P_{-1/2+i\tau}(\cosh\alpha)\sinh\alpha\,d\alpha \tag{1.7}$$

Let us substitute $f(\tau)$ from (1.7) into the first equation of (1.1). After some transformations we obtain Abel's equation

$$\int_s^a \frac{V(\alpha)\sinh\alpha\,d\alpha}{\sqrt{\cosh\alpha - \cosh s}} = G'(0, s) - H(a, \infty) \tag{1.8}$$

for the determination of $V(\alpha)$. The solutions of (1.8) and (0.3) yield

$$-\pi\sinh z\,V(z) = \frac{d}{dz}\int_z^a \frac{G'(0, s) - H(a, \infty)}{\sqrt{\cosh s - \cosh z}}\sinh s\,ds \tag{1.9}$$

2. Next, we consider the dual integral equations

$$\int_0^\infty \tau f(\tau) P_{-1/2+i\tau}(\cosh\alpha)\, d\tau = g(\alpha) \qquad (0 \leqslant \alpha < a)$$

$$\int_0^\infty \tanh \pi\tau f(\tau) P_{-1/2+i\tau}(\cosh\alpha)\, d\tau = h(\alpha) \qquad (a < \alpha < \infty) \tag{2.1}$$

In a manner analogous to the one of Section 1 we obtain for $f(\tau)$ of the system (2.1) the expression

$$\frac{\pi}{\sqrt{2}} f(\tau) = \int_0^a G(0, s) \sin \tau s\, ds - \int_a^\infty H'(s, \infty) \sin \tau s\, ds \tag{2.2}$$

For the first integral of the system (2.1) in the region $(a < \alpha < \infty)$, and for the second integral in the regior $(0 \leqslant \alpha < a)$, we obtain, respectively:

$$-\pi \sinh z \int_0^\infty \tau f(\tau) P_{-1/2+i\tau}(\cosh z)\, d\tau = \frac{d}{dz} \int_a^z \frac{G(0, a) + H'(s, \infty)}{\sqrt{\cosh z - \cosh s}} \sinh s\, ds \tag{2.3}$$

$$\pi \int_0^\infty \tanh \pi\tau f(\tau) P_{-1/2+i\tau}(\cosh\alpha)\, d\tau = \int_\alpha^a \frac{G(0, s)\, ds}{\sqrt{\cosh s - \cosh\alpha}} - \int_a^\infty \frac{H'(s, \infty)\, ds}{\sqrt{\cosh s - \cosh\alpha}} \tag{2.4}$$

3. Let us consider the dual integral equations

$$\int_0^\infty \tau^k f(\tau) \sin \tau s\, d\tau = g(s) \qquad (0 \leqslant s < a)$$

$$(k = \pm 1) \tag{3.1}$$

$$\int_0^\infty \coth \pi\tau f(\tau) \sin \tau s\, d\tau = h(s) \qquad (a < s < \infty)$$

which contain trigonometric functions.

We differentiate, with respect to s, the first equation (3.1) when $k = -1$; when $k = +1$ we integrate this equation with respect to s from 0 to s

Let us introduce the function

$$g_1(s) = g'(s) \quad \text{for } k = -1, \qquad g_1(s) = -\int_0^s g(s)\, ds + C \quad \text{for } k = +1 \tag{3.2}$$

Equations (3.1) can be written in the form

$$\int_0^\infty f(\tau) \cos \tau s\, d\tau = g_1(s) \qquad (0 \leqslant s < a)$$

$$\int_0^\infty \coth \pi\tau f(\tau) \sin \tau s\, d\tau = h(s) \qquad (a < s < \infty) \tag{3.3}$$

Let is multiply the first equation of (3.3) by $\sqrt{2}\pi^{-1}(\cosh\alpha - \cosh s)^{-\frac{1}{2}}$

and integrate it with respect to s from 0 to α. The second equation of (3.3) we multiply by $\sqrt{2}\pi^{-1}(\cosh s - \cosh \alpha)^{-\frac{1}{2}}$ and integrate with respect to s from α to ∞. Using the formulas for the integral representation of the conical functions (0.1), we can reduce the system (3.3) to the form

$$\int_0^\infty f(\tau)\, P_{-1/2+i\tau}(\cosh\alpha)\, d\tau = \Omega(\alpha) \qquad (0 \leqslant \alpha < a) \tag{3.4}$$

$$\int_0^\infty f(\tau)\, P_{-1/2+i\tau}(\cosh\alpha)\, d\tau = \omega(\alpha) \qquad (a < \alpha < \infty)$$

where

$$\Omega(\alpha) = \frac{\sqrt{2}}{\pi}\int_0^\alpha \frac{g_1(s)\, ds}{\sqrt{\cosh\alpha - \cosh s}}, \qquad \omega(\alpha) = \frac{\sqrt{2}}{\pi}\int_\alpha^\infty \frac{h(s)\, ds}{\sqrt{\cosh s - \cosh\alpha}} \tag{3.5}$$

Making use of the formulas for the inversion of the Mehler-Fock transform (0.2), we obtain from (3.4)

$$f(\tau) = \tau \tanh \pi\tau \left[\int_0^a \Omega(\alpha)\, P_{-1/2+i\tau}(\cosh\alpha) \sinh\alpha\, d\alpha + \int_a^\infty \omega(\alpha)\, P_{-1/2+i\tau}(\cosh\alpha) \sinh\alpha\, d\alpha\right] \tag{3.6}$$

Substituting $f(\tau)$ from (3.6) into (3.1) and taking into consideration (0.4), we can derive expressions for the integrals

$$\sqrt{2}\int_0^\infty \tau^{-1} f(\tau) \sin\tau s\, d\tau = \int_0^a \frac{\Omega(\alpha)\sinh\alpha\, d\alpha}{\sqrt{\cosh s - \cosh\alpha}} + \int_a^s \frac{\omega(\alpha)\sinh\alpha\, d\alpha}{\sqrt{\cosh s - \cosh\alpha}}$$

$$-\sqrt{2}\int_0^\infty \coth \pi\tau f(\tau) \sin\tau s\, d\tau = \frac{d}{ds}\left[\int_s^a \frac{\Omega(\alpha)\sinh\alpha\, d\alpha}{\sqrt{\cosh\alpha - \cosh s}} + \int_a^\infty \frac{\omega(\alpha)\sinh\alpha\, d\alpha}{\sqrt{\cosh\alpha - \cosh s}}\right] \tag{3.7}$$

When $k = +1$, the solution of the system (3.1) is given by the formula (3.6), and it can be expressed by means of a constant C which is given by

$$C = \int_0^\infty f(\tau)\, d\tau \tag{3.8}$$

as can easily be verified.

Substituting $f(\tau)$ from (3.6) into (3.8) and solving the obtained relation for C, we obtain its value.

4. In an analogous manner, the solution of the dual integral equations

$$\int_0^\infty f(\tau) \cos\tau s\, d\tau = g_1(s) \qquad (0 \leqslant s < a) \tag{4.1}$$

$$\int_0^\infty \tau^k \coth \pi\tau f(\tau) \cos\tau s\, d\tau = h_1(s) \qquad (a < s < \infty) \qquad (k = \pm 1)$$

can be obtained in the form (3.6) and (3.5), where

$$h(s) = -h_1'(s) \qquad \text{for } k=-1$$
$$h(s) = -\int_s^\infty h_1(s)\,ds + C_1 \qquad \text{for } k=+1 \tag{4.2}$$

The integrals which appear in (4.1) become

$$\sqrt{2}\int_0^\infty \tau^{-1}\coth\pi\tau f(\tau)\cos\tau s\,d\tau = \int_s^a \frac{\Omega(\alpha)\sinh\alpha\,d\alpha}{\sqrt{\cosh\alpha-\cosh s}} + \int_a^\infty \frac{\omega(\alpha)\sinh\alpha\,d\alpha}{\sqrt{\cosh\alpha-\cosh s}}$$
$$\sqrt{2}\int_0^\infty f(\tau)\cos\tau s\,d\tau = \frac{d}{ds}\left[\int_0^a \frac{\Omega(\alpha)\sinh\alpha\,d\alpha}{\sqrt{\cosh s-\cosh\alpha}} + \int_a^s \frac{\omega(\alpha)\sinh\alpha\,d\alpha}{\sqrt{\cosh s-\cosh\alpha}}\right] \tag{4.3}$$

The unknown constant C_1 is determined in the same way as C is determined in Section 3.

5. All solutions of the dual integral equations considered here are obtained in a formal way. We shall now prove the correctness of these solutions. First we consider Equations (1.1), where for the sake of simplicity we take $h(\alpha) = 0$.

We seek solutions of these equations in the form

$$\tau f(\tau) = \int_0^a G(s)\sin\tau s\,ds = -\tau\int_0^a G_1(s)\cos\tau s\,ds \tag{5.1}$$

where

$$G_1(s) = -\int_s^a G(s)\,ds, \qquad G_1'(s) = G(s) \tag{5.2}$$

Let us express the integrals which appear in (1.1) in terms of the functions $G(s)$ or $G_1(s)$. From (5.1) and (0.4) we obtain

$$\int_0^\infty \tau\tanh\pi\tau\, f(\tau)\,P_{-1/2+i\tau}(\cosh\alpha)\,d\tau = \begin{cases} J(a,\alpha) & (0\leqslant\alpha<a)\\ 0 & (a<\alpha<\infty)\end{cases} \tag{5.3}$$

$$\int_0^\infty f(\tau)\,P_{-1/2+i\tau}(\cosh\alpha)\,d\tau = \begin{cases} J_1(0,\alpha) & (0\leqslant\alpha\leqslant a)\\ -J_1(a,0) & (a\leqslant\alpha<\infty)\end{cases} \tag{5.4}$$

where

$$J(a,\alpha) = \int_\alpha^a \frac{G(s)\,ds}{\sqrt{2(\cosh s-\cosh\alpha)}}, \qquad J_1(a,\alpha) = \int_\alpha^a \frac{G_1(s)\,ds}{\sqrt{2(\cosh\alpha-\cosh s)}} \tag{5.5}$$

From (5.3) it follows that the second equation of (1.1), with $h(\alpha) = 0$, is satisfied identically, and from the first equation of (1.1) and from (5.4) it follows that the function $G_1(s)$ must satisfy Abel's integral equation

$$J_1(a,\alpha) = -g(\alpha) \qquad (0\leqslant\alpha\leqslant a) \tag{5.6}$$

the solution of which has the form

$$-\frac{\pi}{\sqrt{2}}G_1(z) = \frac{d}{dz}\int_0^z \frac{g(\alpha)\sinh\alpha\,d\alpha}{\sqrt{\cosh z-\cosh\alpha}} = \sinh z\left[\frac{g(0)}{\sqrt{\cosh z-1}} + \int_0^z \frac{g'(\alpha)\,d\alpha}{\sqrt{\cosh z-\cosh\alpha}}\right] \tag{5.7}$$

It is easy to see that Formulas (5.1) to (5.7) coincide with Formulas (1.3) to (1.5) and (1.9) when $h(\alpha) = 0$. When $h(\alpha) \neq 0$ the correctness of these formulas and of the remaining solutions of the dual integral equations can be verified in a similar way.

6. In the process of solving problems of the theory of elasticity in toroidal coordinates, one encounters dual integral equations of the following type:

$$\begin{aligned}&\int_0^\infty f(\tau)\,[1+N(\tau)]\,P_{-1/2+i\tau}(\cosh\alpha)\,d\tau = g(\alpha) \qquad (0\leqslant\alpha<a)\\ &\int_0^\infty \tau\tanh\pi\tau f(\tau)\,P_{-1/2+i\tau}(\cosh\alpha)\,d\tau = 0 \qquad (a<\alpha<\infty)\end{aligned} \tag{6.1}$$

$$\begin{aligned}&\int_0^\infty \tau f(\tau)\,P_{-1/2+i\tau}(\cosh\alpha)\,d\tau = 0 \qquad (0\leqslant\alpha<a)\\ &\int_0^\infty \tanh\pi\tau f(\tau)\,[1+N(\tau)]\,P_{-1/2+i\tau}(\cosh\alpha)\,d\tau = g(\alpha) \qquad (a<\alpha<\infty)\end{aligned} \tag{6.2}$$

Here the function $N(\tau)$ is assumed to be absolutely integrable.

We will express the solution of the dual equation (6.1) in the form (5.1) and (5.2). Then, in view of (5.3), the second equation of (6.1) will be satisfied identically. Using Formulas (0.1), (0.4) and (5.1), we obtain for the first integral of (6.1) the expression

$$\begin{aligned}&-\int_0^\infty f(\tau)\,[1+N(\tau)]\,P_{-1/2+i\tau}(\cosh\alpha)\,d\tau =\\ &=\frac{\sqrt{2}}{\pi}\int_0^\alpha \frac{dx}{\sqrt{\cosh\alpha-\cosh x}}\int_0^a K(x,s)\,G_1(s)\,ds + \begin{cases}-J_1(0,\alpha) & (0\leqslant\alpha\leqslant a)\\ J_1(a,0) & (a\leqslant\alpha<\infty)\end{cases}\end{aligned} \tag{6.3}$$

where

$$K(x,s)=\int_0^\infty N(\tau)\cos\tau s\cos\tau x\,d\tau \tag{6.4}$$

From the first equation of (6.1), from (6.3) and (5.5) we obtain Abel's integral equation

$$\int_0^\alpha \frac{dx}{\sqrt{\cosh\alpha-\cosh x}}\left[\frac{G_1(x)}{\sqrt{2}}+\frac{\sqrt{2}}{\pi}\int_0^a K(x,s)\,G_1(s)\,ds\right] = -g(\alpha)$$

From this and from (0.3) one obtains, for the determunation of the unknown function $G_1(x)$, the Fredholm integral equation of the second kind,

$$G_1(x)+\frac{2}{\pi}\int_0^a K(x,s)G_1(s)\,ds = -\frac{\sqrt{2}}{\pi}\sinh x\left[\frac{g(0)}{\sqrt{\cosh x-1}}+\int_0^x\frac{g'(\alpha)\,d\alpha}{\sqrt{\cosh x-\cosh\alpha}}\right] \tag{6.5}$$

the kernel of which is a continuous symmetric function of its arguments. It is easy to show that the number $2/\pi$ is not a characteristic value of the kernel (6.4).

If the solution of the dual equations (6.2) is sought in the form

$$\tau f(\tau)=\int_a^\infty H(s)\cos\tau s\,ds = \tau\int_a^\infty H_1(s)\sin\tau s\,ds, \qquad H_1(s)=\int_a^s H(s)\,ds \tag{6.6}$$

then the following integral equation for $H_1(s)$ is found in an analogous manner

$$H_1(x) + \frac{2}{\pi}\int_a^\infty K_1(x, s)\, H_1(s)\, ds = -\frac{\sqrt{2}}{\pi}\frac{d}{dx}\int_x^\infty \frac{g(\alpha)\sinh\alpha\, d\alpha}{\sqrt{\cosh\alpha - \cosh x}} \qquad (6.7)$$

Here

$$K_1(x, s) = \int_0^\infty N(\tau)\sin\tau x \sin\tau s\, d\tau \qquad (6.8)$$

7. As an example, let us consider the torsion problem of a sperical segment when the torsion is caused by rotating a small circular disk fastened rigidly at the center of the flat part of the bounding plane. The spherical part of the surface is fixed (Fig.1).

The remaining part of the flat boundary of the spherical segment is considered free of any external loading, for simplicity sake.

In toroidal coordinates (α, β, φ) (*)

$$r = \sqrt{x^2 + y^2} = \frac{c\sinh\alpha}{\cosh\alpha + \cos\beta}, \qquad z = \frac{c\sin\beta}{\cosh\alpha + \cos\beta} \qquad (c > 0) \qquad (7.1)$$

the given problem can be reduced to the integration of the Michel's equation

$$\frac{\partial^2\Psi}{\partial\alpha^2} + \frac{\partial^2\Psi}{\partial\beta^2} + \frac{3(\cosh\alpha\cos\beta + 1)}{\sinh\alpha(\cosh\alpha + \cos\beta)}\frac{\partial\psi}{\partial\alpha} + \frac{3\sin\beta}{\cosh\alpha + \cos\beta}\frac{\partial\psi}{\partial\beta} = 0 \qquad (7.2)$$

under the following boundary conditions ($\varkappa$ is the angle of rotation of the disk)

$$v(\alpha, 0) = \varkappa r\ (0 \leqslant \alpha < a), \quad \tau_{\beta\varphi}(\alpha, 0) = 0\ (a < \alpha < \infty), \quad v(\alpha, \beta_1) = 0\ (0 \leqslant \alpha < \infty) \quad (7.3)$$

The stresses $\tau_{\alpha\varphi}$ and $\tau_{\beta\varphi}$ and the displacements v are expressed in terms of the displacement function $\Psi(\alpha, \beta)$ by means of Formulas

$$\tau_{\alpha\varphi} = G\sinh\alpha\frac{\partial\Psi}{\partial\alpha}, \qquad \tau_{\beta\varphi} = G\sinh\alpha\frac{\partial\Psi}{\partial\beta}, \qquad v = \frac{c\sinh\alpha}{\cosh\alpha + \cos\beta}\Psi(\alpha, \beta) \qquad (7.4)$$

We look for a solution of Equation (7.2) in the form of an integral of the type

$$\Psi(\alpha, \beta) = \frac{(\cosh\alpha + \cos\beta)^{3/2}}{\sinh\alpha}\int_0^\infty f(\tau)\tanh\pi\tau\frac{\sinh\tau(\beta_1 - \beta)}{\cosh\tau\beta_1}P^1_{-1/2+i\tau}(\cosh\alpha)\, d\tau \qquad (7.5)$$

where $P^1_{-1/2+i\tau}(\cosh\alpha)$ is an associated Legendre function.

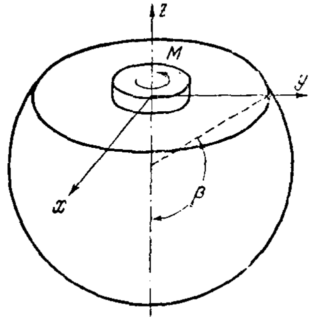

Fig. 1

If one selects (7.5), the last condition of (7.3) is satisfied identically. By making use of the first two conditions of (7.3) for the determination of the unknown function $f(\tau)$ which appears in (7.5), we obtain the dual integral equations

$$\int_0^\infty f(\tau)\tanh\pi\tau\tanh\beta_1\tau P^1_{-1/2+i\tau}(\cosh\alpha)\, d\tau = \frac{\varkappa\sinh\alpha}{(\cosh\alpha + 1)^{3/2}}$$

$$(0 \leqslant \alpha < a) \qquad (7.6)$$

$$\int_0^\infty \tau f(\tau)\tanh\pi\tau\, P^1_{-1/2+i\tau}(\cosh\alpha)\, d\tau = 0 \qquad (a < \alpha < \infty)$$

With the aid of the results of Section 6 and

*) The coordinates are described in detail in the book of Ufland [12].

Formula [12],

$$P^{1}_{-1/2+i\tau}(\cosh\alpha) = \frac{d}{d\alpha} P_{-1/2+i\tau}(\cosh\alpha)$$

the solution of the dual integral equations (7.6) can be reduced to the determination of the function $G_1(x)$ from the Fredholm integral equation of the second kind with a symmetric kernel

$$G_1(x) - \frac{2}{\pi}\int_0^a K(x, s)\, G_1(s)\, ds = -\frac{2}{\pi}\cosh\frac{x}{2}\left[g(0) + k\tanh^2\frac{x}{2}\right] \qquad (7.7)$$

where

$$K(x, s) = \int_0^\infty \frac{\cosh(\pi - \beta_1)}{\cosh \pi\tau \cosh \beta_1 \tau} \cos\tau s \cos\tau x\, d\tau \qquad (7.8)$$

When $\beta_1 = \frac{1}{2}\pi$ (the case of a hemisphere), and when $\beta_1 = \pi$ (the case of a halfspace) the kernel of the integral equation (7.7) takes on the following forms, respectively,

$$K(x, s) = \frac{\cosh {}^1/_2\, x \cosh {}^1/_2 s}{\cosh x + \cosh s}$$

$$K(x, s) = \frac{1}{4\pi}\left[\frac{s + x}{\sinh {}^1/_2 (s + x)} + \frac{s - x}{\sinh {}^1/_2 (s - x)}\right]$$

The unknown constant $g(0)$, which enters into (7.7), will be determined from the condition of boundedness of the sum of shear stresses $\tau_{\beta\varphi}$, which act below the circular disk, i.e by the same method by which the constant C_0 was determined in the paper [13].

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